Dr. rer.nat. Julian Köllermeier, Rik Verbiest, M.Sc., Maddalena Dona, M.Sc.

Differential Equations In Science And Engineering | 22/23 IIa
Sample Solution Examination | 26.06.2023

Exercise 1.

Let $k(t)$ be the number of kangaroos in Australia at time t (the prey) and $p(t)$ the number of predators at time t. This prey and predator situation can be described by the system

$$\begin{aligned} \frac{dk}{dt} &= \alpha k - \beta k^2 - \gamma kp, \\ \frac{dp}{dt} &= -\sigma p + \lambda kp, \end{aligned}$$

where $\alpha, \beta, \gamma, \sigma, \lambda \in \mathbb{R}_+$ are non-negative constants.

a) What are the physical interpretations of the constants $\alpha, \beta, \gamma, \sigma, \lambda$?

b) From now on we assume that $\beta = 0$. What are the two steady states of the ODE system?

c) Compute the stability properties of the steady states of the system. Classify the steady states.

d) Around the non-trivial steady-state, the solution behaves in cycles. Let us assume that the period of such a cycle is T. Show that the average population $\overline{k}, \overline{p}$ during one cycle, defined as

$$\overline{k} = \frac{1}{T}\int_0^T k(t)\,dt, \quad \overline{p} = \frac{1}{T}\int_0^T p(t)\,dt$$

is given by the non-trivial steady-state itself.

What does that mean for the application?

0.5+1+1.5+2 points

Solution.

a)
- α is the specific natural reproduction rate of prey
- β is α divided by the carrying capacity
- γ is the hunting rate
- σ is the death rate of predators
- λ is the birth rate of predators

b) The condition for steady state is that temporal derivatives vanish.

We get from $\frac{dk}{dt} = 0$ and $\frac{dp}{dt} = 0$, that the following three steady states exist:

(1) $k = 0$ and $p = 0$,

(2) $k = \frac{\sigma}{\lambda}$ and $p = \frac{\alpha}{\gamma}$.

c) We compute the Jacobian

$$\begin{pmatrix} \alpha - \gamma p & -\gamma k \\ \lambda p & -\sigma + \lambda k \end{pmatrix}.$$

For stability, we compute the Jacobian at the steady states and compute their eigenvalues, which characterize the stability:

(1) $\begin{pmatrix} \alpha & 0 \\ 0 & -\sigma \end{pmatrix} \Longrightarrow$ EV: $\lambda_1 = \alpha > 0, \lambda_2 = -\sigma < 0$,

(2) $\begin{pmatrix} 0 & -\gamma\sigma/\gamma \\ -\frac{\lambda\alpha}{\gamma} & 0 \end{pmatrix} \Longrightarrow$ EV: $\lambda_{1,2} = \ldots$.

We conclude the following stability properties for the steady states:

(1) (unstable) saddle point,

(2) semi-stable/neutral stable cycle.

d) See lecture notes, Chapter 4.2.4

Exercise 2.

We consider the following system (1) of chemical reactions for the four species A, B, C, D:

$$A + B \xrightarrow{k_1} B + C$$
$$B + C \xrightarrow{k_2} 2B \quad (1)$$
$$C \xrightarrow{k_3} D$$

a) Derive the corresponding system of ODEs that describes the dynamics of the species' concentrations denoted by A, B, C, D.

b) Draw the reaction network.

c) Show that $A + B + C + D$ is a conserved quantity.

2.5+1+1.5 points

Solution.

a) We have 4 substances A, B, C, D, so $N = 4$ and 3 reactions, so $M = 3$.

The stoichiometric coefficients are given by the following table:

$\gamma_{i,m}$	1	2	3
A	-1	0	0
B	0	1	0
C	1	-1	-1
D	0	0	1

The reaction rates are given by

$$\begin{aligned} \lambda_1 &= k_1(T) \cdot n_A \cdot n_B \\ \lambda_2 &= k_2(T) \cdot n_B \cdot n_C \\ \lambda_3 &= k_3(T) \cdot n_C \end{aligned}$$

The production rates R_i of the substances are then given by $R_i = \sum_{m=1}^{3} \gamma_{i,m}\lambda_m$, which means

$$\begin{aligned} R_A &= -\lambda_1 \\ R_B &= +\lambda_2 \\ R_C &= +\lambda_1 - \lambda_2 - \lambda_3 \\ R_D &= +\lambda_3 \end{aligned}$$

This leads to the following ODE system

$$\begin{aligned} \frac{dn_A}{dt} &= -k_1(T)n_A n_B \\ \frac{dn_B}{dt} &= +k_2(T)n_B n_C \\ \frac{dn_C}{dt} &= +k_1(T)n_A n_B - k_2(T)n_B n_C - k_3(T)n_C \\ \frac{dn_D}{dt} &= +k_3(T)n_C. \end{aligned}$$

b) The reaction network is given by

Full points were only given if it is clear where the reaction happens (indicated by the dots in Figure 1). Otherwise the reaction network would not be generalizable for autocatalytic reactions that have product species.

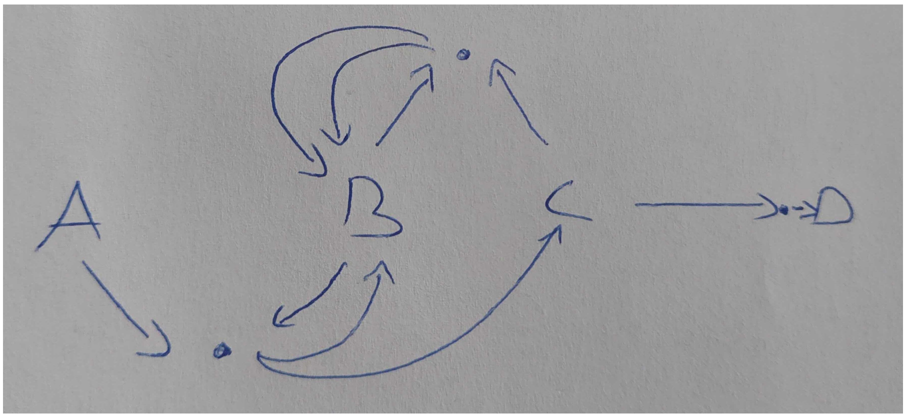

Abbildung 1: Exercise 2 reaction network.

c) For conservation, we solve $\sum_{i=1}^{N} \alpha_i \gamma_{i,m} = 0$ for $m = 1, 2, 3$. This leads to

$$\begin{aligned} -\alpha_1 + \alpha_3 &= 0 \\ +\alpha_2 - \alpha_3 &= 0 \\ -\alpha_3 + \alpha_4 &= 0. \end{aligned}$$

We end up with $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4$. This means that $n_A + n_B + n_C + n_D$ is constant.

Exercise 3.

We consider the scalar wave equation

$$\frac{\partial^2}{\partial t^2}u - c\frac{\partial^2}{\partial x^2}u = 0, \quad c \in \mathbb{R}, \tag{2}$$

with constant wave velocity c.

We want to perform a linear stability analysis of equation (2) using the wave ansatz

$$u(t,x) = c \cdot e^{i(kx-\omega t)}, \tag{3}$$

for wave number $k \in \mathbb{R}$, wave frequencies $\omega \in \mathbb{C}$ and amplitude $c \in \mathbb{R}$.

a) What wave frequencies ω in (3) lead to a stable wave in time?

b) Insert the wave ansatz (3) into the wave equation (2) to derive a stability condition for the wave equation. Show that the stability condition is equivalent to $c > 0$.

c) What is the physical interpretation of this stability condition and does it make sense?

1+3+1 points

Solution.

a) Stability in time means that the wave solution does not increase with t.

Using $\omega = Re(\omega) + iIm(\omega))$, we get that

$$i(-\omega t) = -i(Re(\omega) + iIm(\omega))t = -iRe(\omega)t + Im(\omega)t.$$

The term $-iRe(\omega)t$ is an oscillation in time and not increasing with t. The term $Im(\omega)t$ is potentially increasing and has to be smaller than 0 for stability.

This means that $\omega \in \mathbb{C}$ has to have negative imaginary part: $Im(\omega) < 0$.

b) We use that the derivatives of the wave ansatz (3) are

$$\begin{aligned} \frac{\partial}{\partial t}u &= -\omega i u, \\ \frac{\partial^2}{\partial t^2}u &= \omega^2 u, \\ \frac{\partial}{\partial x}u &= kiu, \\ \frac{\partial^2}{\partial x^2}u &= -k^2 u. \end{aligned}$$

Inserting this into the wave equation (2) yields

$$\begin{aligned} \frac{\partial^2}{\partial t^2}u - c\frac{\partial^2}{\partial x^2}u &= 0 \\ \Rightarrow (-\omega i)^2 u + ck^2 u &= 0 \\ \Rightarrow (ck^2 - \omega^2)u &= 0. \end{aligned}$$

Non-trivial solutions are given by

$$ck^2 - \omega^2 = 0,$$

which leads to

$$\omega = \pm\sqrt{ck^2}. \tag{4}$$

Using the solution of part a), stable solutions are given by $Im(\omega) < 0$. From (4) we see that there will always be a positive and negative solution so the imaginary part needs to be zero overall. This requires

$$c > 0.$$

c) We have $c > 0$. This means that the wave velocity needs to be positive. Waves need to move forward in time, not backward. For positive wave velocity $c > 0$, there will be two waves with wave speeds $\omega = \pm\sqrt{ck^2}$, moving to the left and right, respectively. This is physical for traveling waves.

Exercise 4.

The shallow water equations for water height $h(t,x)$ and vertical velocity $u(t,x)$ are

$$\partial_t \begin{pmatrix} h \\ hu \end{pmatrix} + \partial_x \begin{pmatrix} hu \\ hu^2 + \cos(\alpha) g \frac{h^2}{2} \end{pmatrix} = -\frac{1}{\lambda} \begin{pmatrix} 0 \\ u \end{pmatrix}, \tag{5}$$

where $h(t,x)$ and $u(t,x)$ are the unknowns and g, λ are parameters.

a) What physical interpretations do the equations (5) have and what are the main assumptions for their derivation?

b) Show that the system (5) can be written in the following (so-called primitive variable) form:

$$\partial_t \begin{pmatrix} h \\ u \end{pmatrix} + \begin{pmatrix} u & h \\ \cos(\alpha) g & u \end{pmatrix} \cdot \partial_x \begin{pmatrix} h \\ u \end{pmatrix} = -\frac{1}{\lambda h} \begin{pmatrix} 0 \\ u \end{pmatrix}, \tag{6}$$

c) Assume the spatially homogeneous case in which all spatial derivatives vanish, i.e. $\partial_x h = 0$, $\partial_x u = 0$. Compute the solution of the shallow water equations (5).

d) What problems can appear for numerical schemes trying to solve the homogeneous shallow water equations?

1.5+1.5+1.5+0.5 points

Solution.

a) The first equation is derived from the vertical average of the conservation of mass (also-called continuity equation). The second equation is derived from the vertical average of the conservation of momentum in x-direction.

The main assumptions for the derivation are:

- incompressibility or $\rho = const.$
- shallowness or $\frac{H}{L} = \epsilon \ll 1$.
- plane with inclination angle α.
- friction is modeled using a slip law at the bottom with slip length $\lambda > 0$ (that leads to a relaxation of u towards zero with relaxation time $\frac{1}{\lambda}$).
- gravity is modeled using a gravity constant gravitational acceleration g.

b) The first equation of (6) is derived from the first equation of (5).

The second equation of (6) is derived from the second equation of (5) using the first equation obtained before.

c) The height is constant and the velocity is decaying exponentially.

d) A small value for $h\lambda$ leads to stiffness, which means that the system requires a small time step for a standard explicit scheme.